A SUBGROUP OF A DIRECT PRODUCT OF FREE GROUPS WHOSE DEHN FUNCTION HAS A CUBIC LOWER BOUND

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Abstract. We establish a cubic lower bound on the Dehn function of a certain finitely presented subgroup of a direct product of 3 free groups.

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1. Introduction

The collection $\mathcal{S}$ of subgroups of direct products of free groups is surprisingly rich and has been studied by many authors. In the early 1960s Stallings [13] exhibited a subgroup of $(F_2)^3$, where F_2 is the rank 2 free group, as the first known example of a finitely presented group whose third integral homology group is not finitely generated. Bieri [2] demonstrated that Stallings' group belongs to a sequence of groups $\mathrm{SB}_n \leq (F_2)^n$, the Stallings-Bieri groups, with SB_n being of type $\mathcal{F}_{n-1}$ but not of type $\mathcal{FP}_n$. (See [6] for definitions and background concerning finiteness properties of groups.)

In the realm of decision theory, Mihaĭlova [9] and Miller [10] exhibited a finitely generated subgroup of $(F_2)^2$ with undecidable conjugacy and membership problems. It is thus seen that even in the 2-factor case fairly wild behaviour is encountered amongst the subgroups of direct products of free groups.

In contrast to this, Baumslag and Roseblade [1] showed that in the 2-factor case this wildness only manifests itself amongst the subgroups which are not finitely presented. They proved that if G is a finitely presented subgroup of $F^{(1)} \times F^{(2)}$, where $F^{(1)}$ and $F^{(2)}$ are free groups, then G is itself virtually a direct product of at most 2 free groups. This work was extended by Bridson, Howie, Miller and Short [5] to an arbitrary number of factors. They proved that if G is a subgroup of a direct product of n free groups and if G enjoys the finiteness property $\mathcal{FP}_n$, then G is virtually a direct product of at most n free groups. Further information on the finiteness properties of the groups in class $\mathcal{S}$ was provided by Meinert [8] who calculated the BNS invariants of direct products of free groups. These invariants determine the finiteness properties of all subgroups lying above the commutator subgroup.

Several authors have investigated the isoperimetric behaviour of the finitely presented groups in $\mathcal{S}$. Elder, Riley, Young and this author [7] have shown that the

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Dehn function of Stallings' group SB_3 is quadratic. The method espoused by Bridson in [3] proves that the function n^3 is an upper bound on the Dehn functions of each of the Stallings-Bieri groups. In contrast, there have been no results which give non-trivial lower bounds on the Dehn functions of any groups in $\mathcal{S}$. In other words, until now there has been no finitely presented subgroup of a direct product of free groups whose Dehn function was known to be greater than that of the ambient group. The purpose of this paper is to construct such a subgroup: we exhibit a finitely presented subgroup of $(F_2)^3$ whose Dehn function has the function n^3 as a lower bound.

Let K be the kernel of a homomorphism $\theta : (F_2)^3 \to \mathbb{Z}^2$ whose restriction to each factor F_2 is surjective. By Lemma 3.1 below the isomorphism class of K is independent of the homomorphism chosen. Results in [8] show that K is finitely presented.

Theorem 1.1. *The Dehn function δ of K satisfies $\delta(n) \succeq n^3$.*

Note that Theorem 1.1 makes no reference to a specific presentation of K since, as is well known, the Dehn function of a group is independent (up to $\simeq$-equivalence) of the presentation chosen. We refer the reader to Section 2 for background on Dehn functions, including definitions of the symbols $\succeq$ and $\simeq$.

The organisation of this paper is as follows. Section 2 gives basic definitions concerning Dehn functions, van Kampen diagrams and Cayley complexes. We expect that the reader will already be familiar with these concepts; the purpose of the exposition is principally to introduce notation. In Section 3 we define a class of subgroups of direct products of free groups of which K is a member. The subsequent section gives finite generating sets for those groups in the class which are finitely generated and Section 5 proves a splitting theorem which shows how certain groups in the class decompose as amalgamated products. We then prove in Section 6 how, in certain circumstances, the distortion of the subgroup H in an amalgamated product $\Gamma = G_1 *_H G_2$ gives rise to a lower bound on the Dehn function of Γ. Theorem 1.1 follows as a corollary of this result when applied to the splitting of K given in Section 5.

2. Dehn Functions

In this section we recall the basic definitions concerning Dehn functions of finitely presented groups. All of this material is standard. For further background and a more thorough exposition, including proofs, see, for example, [4] or [11].

Given a set $\mathcal{A}$, write $\mathcal{A}^{-1}$ for the set of formal inverses of the elements of $\mathcal{A}$ and write $\mathcal{A}^{\pm 1}$ for the set $\mathcal{A} \cup \mathcal{A}^{-1}$. Denote by $\mathcal{A}^{\pm *}$ the free monoid on the set $\mathcal{A}^{\pm 1}$. We refer to elements of $\mathcal{A}^{\pm *}$ as words in the letters $\mathcal{A}^{\pm 1}$ and write $|w|$ for the length of such a word w. Given words $w_1, w_2 \in \mathcal{A}^{\pm *}$ we write $w_1 \overset{\text{free}}{=} w_2$ if w_1 and w_2 are freely equal and $w_1 \equiv w_2$ if w_1 and w_2 are equal as elements of $\mathcal{A}^{\pm *}$.

Definition 2.1. Let $\mathcal{P} = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ be a finite presentation of a group G. A word $w \in \mathcal{A}^{\pm *}$ is said to be *null-homotopic* over $\mathcal{P}$ if it represents the identity in G. A *null-$\mathcal{P}$-expression* for such a word is a sequence $(x_i, r_i)_{i=1}^m$ in $\mathcal{A}^{\pm *} \times \mathcal{R}^{\pm 1}$ such that

$$w \overset{\text{free}}{=} \prod_{i=1}^{m} x_i r_i x_i^{-1}.$$

Define the *area* of a null-$\mathcal{P}$-expression to be the integer m and define the $\mathcal{P}$-*area* of w, written $\mathrm{Area}_{\mathcal{P}}(w)$, to be the minimal area taken over all null-$\mathcal{P}$-expressions for w.

The Dehn function of the presentation $\mathcal{P}$, written $\delta_{\mathcal{P}}$, is defined to be the function $\mathbb{N} \to \mathbb{N}$ given by

$$\delta_{\mathcal{P}}(n) = \max\{\mathrm{Area}_{\mathcal{P}}(w) \,:\, w \in \mathcal{A}^{\pm *},\ w \text{ null-homotopic},\ |w| \leq n\}.$$

Although the Dehn functions of different finite presentations of a fixed group may differ, their asymptotic behaviour will be the same. This is made precise in the following way.

Definition 2.2. Given functions $f, g : \mathbb{N} \to \mathbb{N}$, write $f \preceq g$ if there exists a constant $C > 0$ such that $f(n) \leq Cg(Cn + C) + Cn + C$ for all n. Write $f \simeq g$ if $f \preceq g$ and $g \preceq f$.

Lemma 2.3. *If $\mathcal{P}_1$ and $\mathcal{P}_2$ are finite presentations of the same group then $\delta_{\mathcal{P}_1} \simeq \delta_{\mathcal{P}_2}$.*

For a proof of this standard result see, for example, [4, Proposition 1.3.3].

A useful tool for the study of Dehn functions is a class of objects known as van Kampen diagrams. Roughly speaking, these are planar CW-complexes which portray diagrammatically schemes for reducing null-homotopic words to the identity. Such diagrams, whose definition is recalled below, allow the application of topological methods to the calculation of Dehn functions. For background and further details see, for example, [4, Section 4]. For the definition of a combinatorial CW-complex see, for example, [4, Appendix A].

Definition 2.4. A *singular disc diagram* Δ is a finite, planar, contractible combinatorial CW-complex with a specified base vertex $\star$ in its boundary. The *area* of Δ, written $\mathrm{Area}(\Delta)$, is defined to be the number of 2-cells of which Δ is composed. The *boundary cycle* of Δ is the edge loop in Δ which starts at $\star$ and traverses $\partial\Delta$ in the anticlockwise direction. The interior of Δ consists of a number of disjoint open 2-discs, the closures of which are called the *disc components* of Δ.

Each 1-cell of Δ has associated to it two directed edges ϵ_1 and ϵ_2, with $\epsilon_1^{-1} = \epsilon_2$. Let $\mathrm{DEdge}(\Delta)$ be the set of directed edges of Δ. A *labelling* of Δ over a set $\mathcal{A}$ is a map $\lambda : \mathrm{DEdge}(\Delta) \to \mathcal{A}^{\pm 1}$ such that $\lambda(\epsilon^{-1}) = \lambda(\epsilon)^{-1}$. This induces a map from the set of edge paths in Δ to $\mathcal{A}^{\pm *}$. The *boundary label* of Δ is the word in $\mathcal{A}^{\pm *}$ associated to the boundary cycle.

Let $\mathcal{P} = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ be a finite presentation. A $\mathcal{P}$-*van Kampen diagram* for a word $w \in \mathcal{A}^{\pm *}$ is a singular disc diagram Δ labelled over $\mathcal{A}$ with boundary label w and such that for each 2-cell c of Δ the anticlockwise edge loop given by the attaching map of c, starting at some vertex in ∂c, is labelled by a word in $\mathcal{R}^{\pm 1}$.

Lemma 2.5 (Van Kampen's Lemma). *A word $w \in \mathcal{A}^{\pm *}$ is null-homotopic over $\mathcal{P}$ if and only if there exists a $\mathcal{P}$-van Kampen diagram for w. In this case the $\mathcal{P}$-area of w is the minimal area over all $\mathcal{P}$-van Kampen diagrams for w.*

For a proof of this result see, for example, [4, Theorem 4.2.2].

Associated to a presentation $\mathcal{P} = \langle \mathcal{A} \,|\, \mathcal{R} \rangle$ of a group G there is a standard combinatorial 2-complex $K_{\mathcal{P}}$ with $\pi_1(K_{\mathcal{P}}) \cong G$. The complex $K_{\mathcal{P}}$ is constructed by taking a wedge of copies of S^1 indexed by the letters in $\mathcal{A}$ and attaching 2-cells indexed by the relations in $\mathcal{R}$. The 2-cell corresponding to a relation $r \in \mathcal{R}$ has $|r|$

edges and is attached by identifying its boundary circuit with the edge path in $K^1_{\mathcal{P}}$ along which one reads the word r.

The Cayley 2-complex associated to $\mathcal{P}$, denoted by $Cay^2(\mathcal{P})$, is defined to be the universal cover of $K_{\mathcal{P}}$. If one chooses a base vertex of $Cay^2(\mathcal{P})$ to represent the identity element of G then the 1-skeleton of this complex is canonically identified with the Cayley graph of $\mathcal{P}$. Given a $\mathcal{P}$-van Kampen diagram Δ there is a unique label-preserving combinatorial map from Δ to $Cay^2(\mathcal{P})$ which maps the base vertex of Δ to the vertex of $Cay^2(\mathcal{P})$ representing the identity.

3. A Class of Subgroups of Direct Products of Free Groups

In this section we introduce a class of subgroups of direct products of free groups of which the group K defined in the introduction will be a member. We first fix some notation which will be used throughout the paper. Given integers $i, m \in \mathbb{N}$, let $F_m^{(i)}$ be the rank m free group with basis $e_1^{(i)}, \ldots, e_m^{(i)}$. Given an integer $r \in \mathbb{N}$, let $\mathbb{Z}^r$ be the rank r free abelian group with basis $t_1, \ldots, t_r$.

Given positive integers $n, m \geq 1$ and $r \leq m$, we wish to define a group $K_m^n(r)$ to be the kernel of a homomorphism $\theta : F_m^{(1)} \times \ldots \times F_m^{(n)} \to \mathbb{Z}^r$ whose restriction to each factor $F_m^{(i)}$ is surjective. For fixed n, m and r, the isomorphism class of the group $K_m^n(r)$ is, up to an automorphism of the factors of the ambient group $F_m^{(1)} \times \ldots \times F_m^{(n)}$, independent of the homomorphism θ. This is proved by the following lemma.

Lemma 3.1. *Let F be a rank m free group. Given a surjective homomorphism $\phi : F \to \mathbb{Z}^r$, there exists a basis $e_1, \ldots, e_m$ of F so that*

$$\phi(e_i) = \begin{cases} t_i & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } r+1 \leq i \leq m. \end{cases}$$

Proof. The homomorphism ϕ factors through the abelianisation homomorphism $\mathrm{Ab} : F \to A$, where A is the rank m free abelian group $F/[F,F]$, as $\phi = \bar{\phi} \circ \mathrm{Ab}$ for some homomorphism $\bar{\phi} : A \to \mathbb{Z}^r$. Since $\bar{\phi}$ is surjective, A splits as $A_1 \oplus A_2$ where $\bar{\phi}$ is an isomorphism on the first factor and 0 on the second factor. There thus exists a basis $s_1, \ldots, s_m$ for A so as

$$\bar{\phi}(s_i) = \begin{cases} t_i & \text{if } 1 \leq i \leq r, \\ 0 & \text{if } r+1 \leq i \leq m. \end{cases}$$

We claim that the s_i lift under Ab to a basis for F. To see this let $f_1, \ldots, f_m$ be any basis for F and let $\bar{f}_1, \ldots, \bar{f}_m$ be its image under Ab, a basis for A. Let $\rho \in \mathrm{Aut}(A)$ be the change of basis isomorphism from $\bar{f}_1, \ldots, \bar{f}_m$ to $s_1, \ldots, s_m$. It suffices to show that this lifts under Ab to an automorphism of F. But this is certainly the case since $\mathrm{Aut}(A) \cong GL_m(\mathbb{Z})$ is generated by the elementary transformations and each of these obviously lifts to an automorphism. $\square$

Definition 3.2. For integers $n, m \geq 1$ and $r \leq m$, define $K_m^n(r)$ to be the kernel of the homomorphism $\theta : F_m^{(1)} \times \ldots \times F_m^{(n)} \to \mathbb{Z}^r$ given by

$$\theta(e_j^{(i)}) = \begin{cases} t_j & \text{if } 1 \leq j \leq r, \\ 0 & \text{if } r+1 \leq j \leq m. \end{cases}$$

Note that $K_2^n(1)$ is the n^{th} Stallings-Bieri group SB_n.

By a result in Section 1.6 of [8], if $r \geq 1$ and $m \geq 2$ then $K_m^n(r)$ is of type $\mathcal{F}_{n-1}$ but not of type $\mathcal{FP}_n$. In particular the group $K \cong K_2^3(2)$ defined in the introduction is finitely presented.

4. Generating Sets

We give finite generating sets for those groups $K_m^n(r)$ which are finitely generated. We make use of the following notational shorthand: given formal symbols x and y, write $[x, y]$ for $xyx^{-1}y^{-1}$ and x^y for yxy^{-1}.

Proposition 4.1. *If $n \geq 2$ then $K_m^n(r)$ is generated by $S_1 \cup S_2 \cup S_3$ where*

$$S_1 = \{e_i^{(1)}\big(e_i^{(j)}\big)^{-1} : 1 \leq i \leq r, 2 \leq j \leq n\},$$
$$S_2 = \{e_i^{(j)} : r+1 \leq i \leq m, 1 \leq j \leq n\},$$
$$S_3 = \{\big[e_i^{(1)}, e_j^{(1)}\big] : 1 \leq i < j \leq r\}.$$

Proof. Partition S_2 as $S_2' \cup S_2''$ where

$$S_2' = \{e_i^{(j)} : r+1 \leq i \leq m, 2 \leq j \leq n\}$$

and

$$S_2'' = \{e_i^{(1)} : r+1 \leq i \leq m\}.$$

Project $K_m^n(r) \leq F_m^{(1)} \times \ldots \times F_m^{(n)}$ onto the last $n-1$ factors to give the short exact sequence $1 \to K_m^1(r) \to K_m^n(r) \to F_m^{(2)} \times \ldots \times F_m^{(n)} \to 1$. Note that $S_1 \cup S_2'$ projects to a set of generators for $F_m^{(2)} \times \ldots \times F_m^{(n)}$ and that $K_m^1(r)$ is the normal closure in $F_m^{(1)}$ of $S_2'' \cup S_3$. If $\zeta \in F_m^{(1)}$ and $w \equiv w(e_1^{(1)}, \ldots, e_m^{(1)})$ is a word in the generators of $F_m^{(1)}$ then

$$\zeta^w = \zeta^{w\left(e_1^{(1)}(e_1^{(2)})^{-1}, \ldots, e_m^{(1)}(e_m^{(2)})^{-1}\right)}.$$

Thus $S_1 \cup S_2' \cup S_2'' \cup S_3$ generates $K_m^n(r)$. □

5. A Splitting Theorem

The following result gives an amalgamated product decomposition of the groups $K_m^n(r)$ in the case that $r = m$. Note that with slightly more work one could prove a more general result without this restriction. We introduce the following notation: given a collection of groups $M, L_1, \ldots, L_k$ with $M \leq L_i$ for each i, we denote by $\ast_{i=1}^{k}(L_i\,;\,M)$ the amalgamated product $L_1 \ast_M \ldots \ast_M L_k$.

Theorem 5.1. *If $n \geq 2$ and $m \geq 1$ then $K_m^n(m) \cong \ast_{k=1}^{m}(L_k\,;\,M)$, where $M = K_m^{n-1}(m)$ and, for each $k = 1, \ldots, m$, the group $L_k \cong K_m^{n-1}(m-1)$ is the kernel of the homomorphism*

$$\theta_k : F_m^{(1)} \times \ldots \times F_m^{(n-1)} \to \mathbb{Z}^{m-1}$$

given by

$$\theta_k(e_j^{(i)}) = \begin{cases} t_j & \text{if } 1 \leq j \leq k-1, \\ 0 & \text{if } j = k, \\ t_{j-1} & \text{if } k+1 \leq j \leq m. \end{cases}$$

Proof. Projecting $K^n_m(m)$ onto the factor $F^{(n)}_m$ gives the short exact sequence $1 \to K^{n-1}_m(m) \to K^n_m(m) \to F^{(n)}_m \to 1$. This splits to show that $K^n_m(m)$ has the structure of an internal semidirect product $M \rtimes \hat{F}^{(n)}_m$ where $\hat{F}^{(n)}_m \cong F^{(n)}_m$ is the subgroup of $F^{(n-1)}_m \times F^{(n)}_m$ generated by

$$e^{(n-1)}_1 \big(e^{(n)}_1\big)^{-1}, \ldots, e^{(n-1)}_m \big(e^{(n)}_m\big)^{-1}.$$

Since the action by conjugation of $e^{(n-1)}_k \big(e^{(n)}_k\big)^{-1}$ on M is the same as the action of $e^{(n-1)}_k$, we have that

$$\begin{aligned} K^n_m(m) &= M \rtimes \hat{F}^{(n)}_m \\ &\cong \ast^m_{k=1} \Big(M \rtimes \Big\langle e^{(n-1)}_k \big(e^{(n)}_k\big)^{-1} \Big\rangle ;\, M \Big) \\ &\cong \ast^m_{k=1} \Big(M \rtimes \Big\langle e^{(n-1)}_k \Big\rangle ;\, M \Big). \end{aligned}$$

Define a homomorphism $p_k : F^{(1)}_m \times \ldots \times F^{(n-1)}_m \to \mathbb{Z}$ by

$$p_k\big(e^{(i)}_j\big) = \begin{cases} 1 & \text{if } j = k, \\ 0 & \text{otherwise,} \end{cases}$$

and note that $L_k \cap \ker p_k$ is the kernel $K^{n-1}_m(m)$ of the standard homomorphism $\theta : F^{(1)}_m \times \ldots \times F^{(n-1)}_m \to \mathbb{Z}^m$ given in Definition 3.2. Considering the restriction of p_k to L_k gives the short exact sequence $1 \to K^{n-1}_m(m) \to L_k \to \mathbb{Z} \to 1$ which demonstrates that $L_k = K^{n-1}_m(m) \rtimes \langle e^{(n-1)}_k \rangle$. □

6. Dehn Functions of Amalgamated Products

In this section we will be concerned with finitely presented amalgamated products $\Gamma = G_1 \ast_H G_2$ where H, G_1 and G_2 are finitely generated groups and H is proper in each G_i. Suppose each G_i is presented by $\langle \mathcal{A}_i \,|\, \mathcal{R}_i \rangle$, with $\mathcal{A}_i$ finite. Note that we are at liberty to choose the $\mathcal{A}_i$ so as each $a \in \mathcal{A}_i$ represents an element of $G_i \smallsetminus H$. Indeed, since H is proper in G_i there exists some $a' \in \mathcal{A}_i$ representing an element of $G_i \smallsetminus H$ and we can replace each other element $a \in \mathcal{A}_i$ by $a'a$ if necessary.

Let $\mathcal{B}$ be a finite generating set for H and for each $b \in \mathcal{B}$ choose words $u_b \in \mathcal{A}_1^{\pm *}$ and $v_b \in \mathcal{A}_2^{\pm *}$ which equal b in Γ. Define $\mathcal{E} \subset (\mathcal{A}_1 \cup \mathcal{A}_2 \cup \mathcal{B})^{\pm *}$ to be the finite collection of words $\{b u_b^{-1}, b v_b^{-1} : b \in \mathcal{B}\}$. Then, since Γ is finitely presented, there exist finite subsets $\mathcal{R}'_1 \subseteq \mathcal{R}_1$ and $\mathcal{R}'_2 \subseteq \mathcal{R}_2$ such that Γ is finitely presented by

$$\mathcal{P} = \langle \mathcal{A}_1, \mathcal{A}_2, \mathcal{B} \,|\, \mathcal{R}'_1, \mathcal{R}'_2, \mathcal{E} \rangle.$$

Theorem 6.1. *Let $w \in \mathcal{A}_1^{\pm *}$ be a word representing an element $h \in H$ and let $u \in \mathcal{A}_1^{\pm *}$ and $v \in \mathcal{A}_2^{\pm *}$ be words representing elements $\alpha \in G_1 \smallsetminus H$ and $\beta \in G_2 \smallsetminus H$ respectively. If $[\alpha, h] = [\beta, h] = 1$ then*

$$\mathrm{Area}_{\mathcal{P}}([w, (uv)^n]) \geq 2n \, \mathrm{d}_{\mathcal{B}}(1, h)$$

where $\mathrm{d}_{\mathcal{B}}$ is the word metric on H associated to the generating set $\mathcal{B}$.

Proof. Let Δ be a $\mathcal{P}$-van Kampen diagram for the null-homotopic word $[w, (uv)^n]$ (see Diagram 1). For each $i = 1, 2, \ldots, n$, define p_i to be the vertex in $\partial\Delta$ such that the anticlockwise path in $\partial\Delta$ from the basepoint around to p_i is labelled by the word $w(uv)^{i-1}u$. Similarly define q_i to be the vertex in $\partial\Delta$ such that the

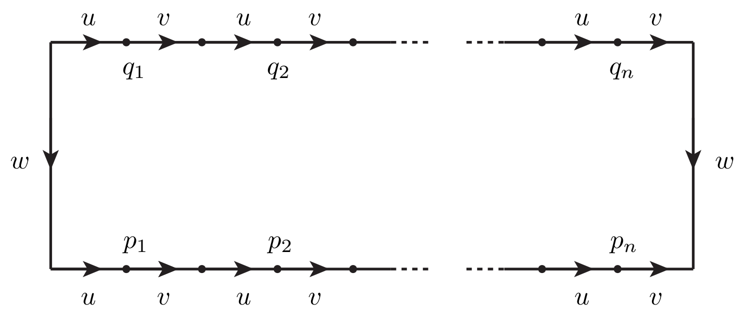

FIGURE 1. The van Kampen diagram Δ

anticlockwise path in $\partial\Delta$ from the basepoint around to q_i is labelled by the word $w(uv)^n w^{-1}(uv)^{i-n}v^{-1}$. We will show that for each i there is a $\mathcal{B}$-path (i.e. an edge path in Δ labelled by a word in the letters $\mathcal{B}$) from p_i to q_i.

We assume that the reader is familiar with Bass-Serre theory, as exposited in [12]. Let T be the Bass-Serre tree associated to the splitting $G_1 *_H G_2$. This consists of an edge gH for each coset Γ/H and a vertex gG_i for each coset Γ/G_i. The edge gH has initial vertex gG_1 and terminal vertex gG_2. We will construct a continuous (but non-combinatorial) map $\Delta \to T$ as the composition of the natural map $\Delta \to Cay^2(\mathcal{P})$ with the map $f : Cay^2(\mathcal{P}) \to T$ defined below.

There is a natural left action of Γ on each of $Cay^2(\mathcal{P})$ and T and we construct f to be equivariant with respect to this as follows. Let m be the midpoint of the edge H of T and define f to map the vertex $g \in Cay^2(\mathcal{P})$ to the point $g \cdot m$, the midpoint of the edge gH. Define f to map the edge of $Cay^2(\mathcal{P})$ labelled $a \in \mathcal{A}_i$ joining vertices g and ga to the geodesic segment joining $g \cdot m$ to $ga \cdot m$. Since $a \notin H$ this segment is an embedded arc of length 1 whose midpoint is the vertex gG_i. Define f to collapse the edge in $Cay^2(\mathcal{P})$ labelled $b \in \mathcal{B}$ joining vertices g and gb to the point $g \cdot m = gb \cdot m$. This is well defined since $gH = gbH$. This completes the definition of f on the 1-skeleton of Δ; we now extend f over the 2-skeleton.

Let c be a 2-cell in $Cay^2(\mathcal{P})$ and let g be some vertex in its boundary. Assume that c is metrised so as to be convex and let l be some point in its interior. The form of the relations in $\mathcal{P}$ ensures that the boundary label of c is a word in the letters $\mathcal{A}_i \cup \mathcal{B}$ for some i and so every vertex in ∂c is labelled gg' for some $g' \in G_i$. Thus f as so far defined maps ∂c into the ball of radius $1/2$ centred on the vertex gG_i; we extend f to the interior of c by defining it to map the geodesic segment $[l, p]$, where $p \in \partial c$, to the geodesic segment $[gG_i, f(p)]$. This is independent of the vertex $g \in \partial c$ chosen and makes f continuous since geodesics in a tree vary continuously with their endpoints. We now define $\bar{f} : \Delta \to T$ to be the map given by composing f with the label-preserving map $\Delta \to Cay^2(\mathcal{P})$ which sends the basepoint of Δ to the vertex $1 \in Cay^2(\mathcal{P})$.

Since w commutes with u and v we have that $\bar{f}(p_i) = w(uv)^{i-1}u \cdot m = (uv)^{i-1}u \cdot m = \bar{f}(q_i)$; define S to be the preimage under $\bar{f}$ of this point. By construction, the image of the interior of each 2-cell in Δ and the image of the interior of each $\mathcal{A}_i$-edge is disjoint from $\bar{f}(p_i)$. Thus S consists of vertices and $\mathcal{B}$-edges and so finding a $\mathcal{B}$-path from p_i to q_i reduces to finding a path in S connecting these vertices. Let s_i and t_i be the vertices of $\partial\Delta$ immediately preceding and succeeding p_i in the

boundary cycle. Unless $h = 1$, in which case the theorem is trivial, the form of the word $[w, (uv)^n]$, together with the normal form theorem for amalgamated products, implies that all the vertices p_i, s_i and t_i lie in the boundary of the same disc component D of Δ. Furthermore, since u and v are words in the letters $\mathcal{A}_1$ and $\mathcal{A}_2$ respectively, the points $f(s_i)$ and $f(t_i)$ are separated in T by $f(p_i)$. Thus s_i and t_i are separated in D by S and so there exists an edge path γ_i in S from p_i to some other vertex $r_i \in \partial D$. Since γ_i is a $\mathcal{B}$-path it follows that the word labelling the sub-arc of the boundary cycle of Δ from p_i to r_i represents an element of H, and, by considering subwords of $[w, (uv)^n]$, we see that the only possibility is that $r_i = q_i$. Thus, for each $i = 1, \dots, n$, the path γ_i gives the required $\mathcal{B}$-path connecting p_i to q_i. We choose each γ_i to contain no repeated edges.

For $i \neq j$, the two paths γ_i and γ_j are disjoint, since if they intersected there would be a $\mathcal{B}$-path joining p_i to p_j and thus the word labelling the subarc of the boundary cycle from p_i to p_j would represent an element of H. Observe that no two edges in any of the paths $\gamma_1, \dots, \gamma_n$ lie in the boundary of the same 2-cell in Δ since each relation in $\mathcal{P}$ contains at most one occurrence of a letter in $\mathcal{B}$. Because the word labelling $\partial\Delta$ contains no occurrences of a letter in $\mathcal{B}$ the interior of each edge of a path γ_i lies in the interior of Δ and thus in the boundary of two distinct 2-cells. Since each path γ_i contains no repeated edges we therefore obtain the bound $\mathrm{Area}(\Delta) \geq \sum_{i=1}^{n} 2|\gamma_i|$. But the word labelling each γ_i is equal to h in Γ and so the length of γ_i is at least $\mathrm{d}_{\mathcal{B}}(1, h)$ whence we obtain the required inequality. $\square$

We are now in a position to prove the main theorem.

Proof of Theorem 1.1. To avoid excessive superscripts we change notation and write x_i, y_i for the generators of $F_2^{(i)}$, $i = 1, 2$.

By Proposition 4.1 and Theorem 5.1 we have that $K \cong K_2^3(2) \cong L_1 *_M L_2$ where, as subgroups of $F_2^{(1)} \times F_2^{(2)}$, $L_1 = K_2^2(1)$ is generated by $\mathcal{A}_1 = \{x_1x_2^{-1}, y_1, y_2\}$, $L_2 \cong K_2^2(1)$ is generated by $\mathcal{A}_2 = \{x_1, x_2, y_1y_2^{-1}\}$ and $M = K_2^2(2)$ is generated by $\mathcal{B} = \{x_1x_2^{-1}, y_1y_2^{-1}, [x_1, y_1]\}$. To obtain the generating set for L_2 we have here implicitly used the automorphism of $F_2^{(1)} \times F_2^{(2)}$ which interchanges x_i with y_i and realises the isomorphism between L_2 and $K_2^2(1)$.

For $n \in \mathbb{N}$, define h_n to be the element $[x_1^n, y_1^n] \in K_2^2(2)$ and define w_n to be the word $[(x_1x_2^{-1})^n, y_1^n] \in \mathcal{A}_1^{\pm *}$ representing h_n. Note that h_n commutes with both $y_2 \in \mathcal{A}_1$ and $x_2 \in \mathcal{A}_2$ so by Theorem 6.1 the word $[w_n, (y_2x_2)^n]$, which has length $16n$, has area at least $2n\, d_{\mathcal{B}}(1, h_n)$. We claim that $d_{\mathcal{B}}(1, h_n) \geq n^2$.

Suppose that in $F_2^{(1)} \times F_2^{(2)}$ the element h_n is represented by a word $w \equiv w(x_1x_2^{-1}, y_1y_2^{-1}, [x_1, y_1])$ in the generators $\mathcal{B}$. Let k be the number of occurrences of the third variable in the word w. We will show that $k \geq n^2$.

Observe that, as a group element, the word $w(x_1x_2^{-1}, y_1y_2^{-1}, [x_1, y_1])$ is equal to the word $w(x_1, y_1, [x_1, y_1])\, w(x_2^{-1}, y_2^{-1}, 1)$. Thus we have that $[x_1^n, y_1^n]$ is freely equal to $w(x_1, y_1, [x_1, y_1])$ and that $w(x_2^{-1}, y_2^{-1}, 1)$, and thus $w(x_1, y_1, 1)$, is freely equal to the empty word. It follows that $[x_1^n, y_1^n]$ can be converted to the empty word by free expansions, free contractions and deletion of k subwords $[x_1, y_1]$. Hence $[x_1^n, y_1^n]$ is a null-homotopic word over the presentation $\mathcal{P} = \langle x_1, y_1 \mid [x_1, y_1] \rangle$ with $\mathcal{P}$-area at most k. But $\mathcal{P}$ presents the rank 2 free abelian group, and basic results on Dehn functions give that $[x_1^n, y_1^n]$ has area n^2 over this presentation. Thus $k \geq n^2$. $\square$

Note that the above proof also shows that $K_2^2(2)$ has at least quadratic distortion in each of $K_2^2(1)$ and $F_2^{(1)} \times F_2^{(2)}$. In fact it can be shown that the distortion is precisely quadratic.

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